Rayliegh's Method for Solving the Mass Loaded String and Bar Problems

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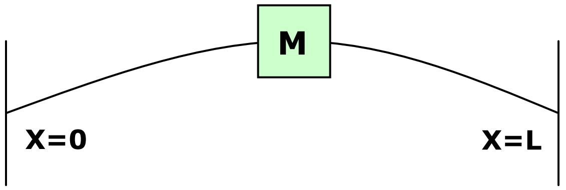


The mass loaded string problem is a good example of a perturbation problem that cannot be easily solved through differential equations. A method that proves very useful in perturbation problems relating to harmonic motion is Rayliegh's Energy Method. Rayliegh's Energy method (at least in one simple form) simply states that:

$$\omega^2 \leq \frac{Potential\ Energy_{\max}}{\frac{1}{\omega^2} Kinetic\ Energy_{\max}}$$

This relation makes the assumption that the potential energy and kinetic energy are equal and there is little energy lost due to friction, damping, etc. One can simply make a guess at a function that satisfies the boundary conditions of the problem and Rayliegh's method will always give a solution for frequency that is greater than or equal to the true result. You could then select a function with parameters that can be tweaked to minimize energy with a fancy algorithm! **The equation above comes from a formalization called the Rayliegh quotient, but you could arrive at it by equating the *maximum* potential and kinetic energy for an *oscillating* system.**

In the mass loaded string problem, the string is fixed at both ends with some mass M in the center (L/2). To approximate the fundamental frequency of vibration, we will use the function for the non-mass loaded string for our 'guess'.

$$y(x,t) = A\sin\left(\pi \frac{x}{L}\right)\sin(\omega t)$$

Now, we just need equations for the energy of the string.

$$P.E. = \frac{T}{2}\int_0^L \left(\frac{dy}{dx}\right)^2 dx$$

$$K.E. = \frac{\rho_l}{2}\int_0^L \left(\frac{dy}{dt}\right)^2 dx$$

Where ρ_l is the linear mass density of the string and *T* is the string tension. These equations were derived by considering a small section of string *dx* and integrating the potential and kinetic energies over the entire length.

Now, let's fill in Rayliegh's equation. First calculate the derivatives:

$$\frac{dy}{dx} = \frac{\pi A}{L}\sin(\omega t)\cos\left(\frac{\pi}{L}x\right)$$

$$\frac{dy}{dt} = A\omega\cos(\omega t)\sin\left(\frac{\pi}{L}x\right)$$

Next, let's get the maximum values of the energies. This will occur when the time dependent part is at a maximum:

$$\frac{dy}{dx}_{\max} = \frac{\pi A}{L}\cos\left(\frac{\pi}{L}x\right)$$

$$\frac{dy}{dt}_{\max} = A\omega\sin\left(\frac{\pi}{L}x\right)$$

Now, fill in Rayliegh's equation:

$$\omega^2 \le \frac{Potential\ Energy_{\max}}{\frac{1}{\omega^2}Kinetic\ Energy_{\max}}$$

$$\omega^2 \le \frac{\frac{T}{2}\int_0^L\left(\frac{\pi A}{L}\cos\left(\frac{\pi}{L}x\right)\right)^2 dx}{\frac{1}{\omega^2}\left[\frac{\rho_l}{2}\int_0^l\left(A\omega\sin\left(\frac{\pi}{L}x\right)\right)^2 dx\right]}$$

Now, you'll notice something missing. We need to account for the mass at the middle of the string. Assuming that it doesn't affect the stiffness at the center of the string, we can treat it as an element that adds to the kinetic energy.

$$K.E._{mass} = \frac{1}{2}M\left(\frac{dy}{dt}\right)^2$$

$$K.E._{mass} = \frac{1}{2}M\left(A\omega\cos(\omega t)\sin\left(\frac{\pi}{L}x\right)\right)^2$$

$$K.E._{mass-\max} = \frac{1}{2}MA^2\omega^2\sin^2\left(\frac{\pi}{L}x\right)$$

Now, Because our mass is at $\frac{L}{2}$, we can complete the equation

$$K.E._{mass-\max} = \frac{1}{2}MA^2\omega^2\sin\left(\frac{\pi}{2}\right)$$

$$K.E._{mass-\max} = \frac{1}{2}MA^2\omega^2$$

Lets add this to our kinetic energy equation in

$$\omega^2 \le \frac{Potential\ Energy_{\max}}{\frac{1}{\omega^2}Kinetic\ Energy_{\max}}$$

$$\omega^2 \le \frac{\frac{T}{2}\int_0^L\left(\frac{\pi A}{L}\cos\left(\frac{\pi}{L}x\right)\right)^2 dx}{\frac{1}{\omega^2}\left[\frac{\rho_l}{2}\int_0^l\left(A\omega\sin\left(\frac{\pi}{L}x\right)\right)^2 dx + \frac{1}{2}MA^2\omega^2\right]}$$

Now it’s just an algebra and calculus problem!

$$\omega^2 \le \frac{\frac{T}{2}\frac{\pi^2A^2}{L^2}\int_0^L\cos^2\left(\frac{\pi}{L}x\right)dx}{\frac{1}{\omega^2}\left[\frac{\rho_l}{2}A^2\omega^2\int_0^l\sin^2\left(\frac{\pi}{L}x\right)dx + \frac{1}{2}MA^2\omega^2\right]}$$

$$\omega^2 \le \frac{T\frac{\pi^2}{L^2}\int_0^L\cos^2\left(\frac{\pi}{L}x\right)dx}{\rho_l\int_0^l\sin^2\left(\frac{\pi}{L}x\right)dx + M}$$

Now, lets integrate the top:

$$\int_0^L\cos^2\left(\frac{\pi}{L}x\right)dx = \frac{x}{2} + \left.\frac{L\sin(\frac{2\pi}{L}x)}{4\pi}\right|_0^L = \frac{L}{2}$$

and now the bottom:

$$\int_0^l \sin^2\left(\frac{\pi}{L}x\right)dx = \left.\frac{x}{2} - \frac{L\sin\left(\frac{2\pi}{L}x\right)}{4\pi}\right|_0^L = \frac{L}{2}$$

Collapse our energy equation:

$$\omega^2 \le \frac{T\frac{\pi^2}{L^2}\int_0^L \cos^2\left(\frac{\pi}{L}x\right)dx}{\rho_l \int_0^l \sin^2\left(\frac{\pi}{L}x\right)dx + M}$$

$$\omega^2 \le \frac{T\frac{\pi^2}{L^2}\frac{L}{2}}{\rho_l\frac{L}{2}+M} \le \frac{\frac{T\pi^2}{2L}}{\left(\rho_l\frac{L}{2}+M\right)}$$

$$\omega^2 \le \frac{T\pi^2}{2L\left(\rho_l\frac{L}{2}+M\right)} \le \frac{T\pi^2}{\left(\rho_l L^2 + 2LM\right)}$$

$$\omega \le \sqrt{\frac{T\pi^2}{\left(\rho_l L^2 + 2LM\right)}}$$

$$f \le \frac{1}{2\pi}\sqrt{\frac{T\pi^2}{\left(\rho_l L^2 + 2LM\right)}} = \frac{1}{2}\sqrt{\frac{T}{\left(\rho_l L^2 + 2LM\right)}}$$

So here is our equation to find the frequency for a center mass loaded string:

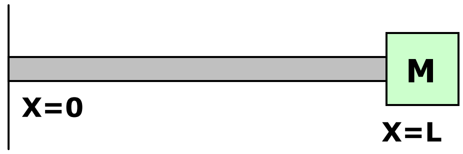
$$f \le \frac{1}{2}\sqrt{\frac{T}{\left(\rho_l L^2 + 2LM\right)}}$$

We can compare this to the frequency of a non-mass loaded fixed-fixed string.

$$f = \frac{1}{2L}\sqrt{\frac{T}{\rho_l}}$$

Notice that when our mass is zero, we get the above equation!

Rayliegh's Method for Solving the Mass Loaded Bar Problem



We can use Rayliegh's method for the mass loaded bar problem as well!

$$\omega^2 \le \frac{Potential\ Energy_{\max}}{\frac{1}{\omega^2} Kinetic\ Energy_{\max}}$$

To approximate the fundamental frequency of vibration, we will use the function for the non-mass loaded bar for our 'guess' function. Note: y represents the displacement of 'particles' of the bar from their initial position. Remember that the restoring force *F* generated in the bar is:

$$F = -SY\frac{\partial y}{\partial x}$$

$$y(x,t) = A\sin\left(\frac{\pi}{2}\frac{x}{L}\right)\sin(\omega t)$$

Energy Relations for the bar are.

$$P.E. = -\frac{1}{2}SY\int_0^L \left(\frac{dy}{dx}\right)^2 dx$$

$$K.E. = \frac{\rho}{2}S\int_0^L \left(\frac{dy}{dt}\right)^2 dx$$

Where *S* is cross sectional area of the bar, ρ is the mass density of the bar and Y is Young's modulus.

Now, find out derivatives:

$$\frac{dy}{dx}_{\max} = \frac{A\pi}{2L}\cos\left(\frac{\pi}{2L}x\right)$$

$$\frac{dy}{dt}_{\max} = A\omega\sin\left(\frac{\pi}{2L}x\right)$$

Plug these into our energy relation:

$$\omega^2 \le \frac{Potential\ Energy_{\max}}{\frac{1}{\omega^2} Kinetic\ Energy_{\max}}$$

$$\omega^2 \le \frac{\frac{SY}{2}\int_0^L \left(\frac{\pi A}{2L}\cos\left(\frac{\pi}{2L}x\right)\right)^2 dx}{\frac{1}{\omega^2}\left[\frac{\rho S}{2}\int_0^l \left(A\omega \sin\left(\frac{\pi}{2L}x\right)\right)^2 dx\right]}$$

Now, let’s add in the additional energy due to the mass

$$\omega^2 \le \frac{\frac{SY}{2}\int_0^L \left(\frac{\pi A}{2L}\cos\left(\frac{\pi}{2L}x\right)\right)^2 dx}{\frac{1}{\omega^2}\left[\frac{\rho_l}{2}\int_0^l \left(A\omega \sin\left(\frac{\pi}{2L}x\right)\right)^2 dx + \frac{1}{2}MA\omega\sin\left(\frac{\pi}{2L}x\right)\right]}$$

$$\omega^2 \le \frac{\frac{SY}{2}\int_0^L \left(\frac{\pi A}{2L}\cos\left(\frac{\pi}{2L}x\right)\right)^2 dx}{\frac{1}{\omega^2}\left[\frac{\rho S}{2}\int_0^l \left(A\omega \sin\left(\frac{\pi}{2L}x\right)\right)^2 dx + \frac{1}{2}MA^2\omega^2\right]}$$

Just do the algebra!

$$\omega^2 \le \frac{\frac{SY}{2}\frac{\pi^2 A^2}{4L^2}\int_0^L \cos^2\left(\frac{\pi}{2L}x\right)dx}{\frac{1}{\omega^2}\left[\frac{\rho S}{2}A^2\omega^2\int_0^l \sin^2\left(\frac{\pi}{2L}x\right)dx + \frac{1}{2}MA^2\omega^2\right]}$$

$$\omega^2 \le \frac{-\frac{SY\pi^2}{4L^2}\int_0^L \cos^2\left(\frac{\pi}{2L}x\right)dx}{\left[\rho S\int_0^l \sin^2\left(\frac{\pi}{2L}x\right)dx + M\right]}$$

The integrals:

$$\int_0^L \cos^2\left(\frac{\pi}{2L}x\right)dx$$

$$\left.\frac{1}{2}x+\frac{2L\sin\left(\frac{\pi}{L}x\right)}{4\pi}\right|_0^L=\frac{L}{2}$$

$$\int_0^L \sin^2\left(\frac{\pi}{2L}x\right)dx$$

$$\left.\frac{1}{2}x-\frac{2L\sin\left(\frac{\pi}{L}x\right)}{4\pi}\right|_0^L=\frac{L}{2}$$

Back to our energy equation:

$$\omega^2\le\frac{\frac{SY\pi^2}{4L^2}\frac{L}{2}}{\left[\rho S\frac{L}{2}+M\right]}\le\frac{\frac{SY\pi^2}{8L}}{\left[\rho SL+M\right]}\le\frac{SY\pi^2}{8L\left[\rho SL+M\right]}$$

After all the work:

$$f\le\frac{1}{4}\sqrt{\frac{SY}{\left[\rho SL^2+2LM\right]}}$$

The frequency for the bar without the mass is

$$f=\frac{1}{4L}\sqrt{\frac{Y}{\rho}}$$

Notice that the mass loaded bar frequency matches when M is zero!

References:

Fundamentals of Acoustics – Lawernce Kinsler, Austin Frey 4th Edition. 2000.

Mechanical Vibrations – Den Hartog. Dover Edition. 1985