# Finite Difference Math

## 1. Difference Operators

$\Delta f_i = f_{i+1} - f_i$ Forward Difference Operator

$\nabla f_i = f_i - f_{i-1}$ Backward Difference Operator

$\delta f_i = f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}$ Central Difference Operator

Note that any of these divided by a discrete change in the independent variable, $h = \Delta x$ approximates the derivative definition:

$$\frac{\partial f}{\partial x} = \lim_{\mathrm{h}\to 0} \frac{f(x+h) - f(x)}{h}$$
$$\cong \frac{\Delta f_i}{h} + \mathrm{O}(h) \cong \frac{\nabla f_i}{h} + \mathrm{O}(h) \cong \frac{\delta f_i}{h} + \mathrm{O}(h^2)$$

Where $\mathrm{O}(h)$ represents higher order terms. Such an approximation is termed a *first order accurate* solution, because the order of the truncation terms, $\mathrm{O}(h)$, is 1. The central difference method is *second order accurate* because it's truncation terms start at $\mathrm{O}(h^2)$. However, the caveat of the centered difference is that we have defined our spatial grid on discrete points, $n\Delta x = n\,h$, not half points, $n\frac{\Delta x}{2}$. In most applications this is not possible and the centered difference method serves only as a more accurate theoretical expression. The centered difference method for the second derivative does exist, and should be used when possible (see below).

Expanding on the first differences, the second forward difference is as follows:

$$\Delta^2 f_i = \Delta(\Delta f_i) = \Delta(f_{i+1} - f_i) = \Delta f_{i+1} - \Delta f_i = f_{i+2} - f_{i+1} - f_{i+1} + f_i$$

$$\Delta^2 f_i = f_{i+2} - 2f_{i+1} + f_i$$

Similarly, for the backwards and centered difference schemes:

$$\nabla^2 f_i = f_i - 2f_i + f_{i-2}$$

$$\delta^2 f_i = f_{i+1} - 2f_i + f_{i-1}$$

Note that the second centered difference (and every even derivative) actually exists in physical space. This is useful. Third Differences are as follows:

$$\Delta^3 f_i = f_{i+3} - 3\,f_{i+2} + 3f_{i+1} - f_i$$

$$\nabla^3 f_i = f_i - 3f_{i-1} + 3f_{i-2} - f_{i-3}$$

$$\delta^3 f_i = f_{i+\frac{3}{2}} - 3f_{i+\frac{1}{2}} + 3f_{i-\frac{1}{2}} - f_{i-\frac{3}{2}}$$

## 2. Differential Operator

Recall the Taylor series expansion:

$$f(x+h) = f(x) + h\frac{df}{dx} + \frac{h^2}{2!}\frac{d^2 f}{dx^2} + \frac{h^3}{3!}\frac{d^3 f}{dx^3} + \cdots$$

Using the difference notation,

$$f_{i+1} = f_i + hf_i^{'} + \frac{h^2}{2!}f_i^{''} + \frac{h^3}{3!}f_i^{'''} + \cdots$$

The *Differential Operator* is mostly defined to make things more neat looking: $D = d/dx$, $D^2 = d^2/dx$^2. Plugging through the math and using the differential operator notation, we can express any derivative of a function numerically with an increasing amount of accuracy depending on the order used. For first-order accuracy, the first and second derivatives and third (and so on…) in the forward difference scheme can be estimated:

$$Df_i = \frac{df_i}{dx} = \frac{1}{h}\left(\Delta f_i - \frac{\Delta^2}{2}f_i + \frac{\Delta^3}{3}f_i - \cdots\right)$$

$$D^2 f_i = \frac{d^2 f_i}{dx^2} = 1/h^2\left(\Delta^2 f_i - \Delta^3 f_i + \frac{11}{12}\Delta^4 f_i - \cdots\right)$$

$$D^3 f_i = \frac{d^3 f_i}{dx^2} = \frac{1}{h^3}\left(\Delta^3 f_i - \frac{3}{2}\Delta^4 f_i + \frac{7}{4}\Delta^5 f_i - \cdots\right)$$

The same powers hold for the backward differencing scheme except:

1) the Δ's are ∇'s
2) All signs are + instead of alternating -,+,-,+,…

For the centered difference method, the second and fourth derivatives are as follows:

$$D^2 f_i = \frac{d^2 f_i}{dx^2} = 1/h^2\left(\delta^2 f_i - \frac{\delta^4}{12} + \frac{\delta^6}{90} - \cdots\right)$$

$$D^4 f_i = \frac{d^4 f_i}{dx^2} = \frac{1}{h^4}\left(\delta^4 - \frac{\delta^6}{6} + \frac{7}{240}\delta^8 - \cdots\right)$$

The first and third centered differentials aren't very realistic because again, they depend on differential operators to the odd derivative which results in taking spatial intervals at $\frac{1}{2\Delta x}$.

# Table of Finite Difference Derivative Expressions

**Taylor Series**

$$f(x+h)=f(x)+h\frac{df}{dx}+\frac{h^2}{2!}\frac{d^2f}{dx^2}+\frac{h^3}{3!}\frac{d^3f}{dx^3}+\cdots$$

| | Forward | Backward | Central |
|---|---|---|---|
| O(h) | $Df_i=\frac{1}{h}(f_{i+1}-f_i)$ | $\frac{1}{h}(f_i-f_{i-1})$ | |
| | $D^2f_i=\frac{1}{h^2}(f_{i+2}-2f_{i+1}+f_i)$ | $\frac{1}{h^2}(f_i-2f_{i-1}+f_{i-2})$ | |
| | $D^3f_i=\frac{1}{h^3}(f_{i+3}-3f_{i+2}+3f_{i+1}-f_i)$ | $\frac{1}{h^3}(f_i-3f_{i-1}+3f_{i-2}-f_{i-3})$ | |
| | $D^4f_i=\frac{1}{h^4}(f_{i+4}-4f_{i+3}+6f_{i+2}-4f_{i+1}+f_i)$ | $\frac{1}{h^4}(f_i-4f_{i-1}+6f_{i-2}-4f_{i-3}+f_{i-4})$ | |
| O(h²) | $Df_i=\frac{1}{2h}(-f_{i+2}+4f_{i+1}-3f_i)$ | $\frac{1}{2h}(3f_i-4f_{i-1}+f_{i-2})$ | $\frac{1}{2h}(f_{i+1}-f_{i-1})$ |
| | $D^2f_i=\frac{1}{h^2}(-f_{i+3}+4f_{i+2}-5f_{i+1})$ | $\frac{1}{h^2}(2f_i-5f_{i-1}+4f_{i-2}-f_{i-3})$ | $\frac{1}{h^2}(f_{i+1}-2f_i+f_{i-1})$ |
| | $D^3f_i=\frac{1}{2h^3}(-3f_{i+4}+14f_{i+3}-24f_{i+2}+18f_{i+1}-5f_i)$ | $\frac{1}{2h^3}(5f_i-18f_{i-1}+24f_{i-2}-14f_{i-3}+3f_{i-4})$ | $\frac{1}{2h^3}(f_{i+2}-2f_{i+1}+2f_{i-1}-f_{i-2})$ |
| | $D^4f_i=\frac{1}{h^4}(-2f_{i+5}+11f_{i+4}-24f_i+3+26f_{i+2}-14f_{i+1}+3f_i$ | $\frac{1}{h^4}(3f_i-14f_{i-1}+26f_{i-2}-24f_{i-3}+11f_{i-4}-2f_{i-5})$ | $\frac{1}{h^4}(f_{i+2}-4f_{i+1}+6f_i-4f_{i-1}+f_{i-2})$ |